Indian Statistical Institute

Midterm Examination 2017-2018

B.Math Third Year   Complex Analysis   September 12, 2017   Instructor : Jaydeb Sarkar

Time : 3 Hours   Maximum Marks : 100

(i) $U \subseteq \mathbb{C}$ open. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $Hol(U) = \{f : U \to \mathbb{C}$ holonorphic $\}$. (v) $\mathcal{D}$ = a domain in $\mathbb{C}$.

(1) (15 marks) Let $f \in Hol(\mathcal{D})$, and let L be a line in $\mathbb{C}$. If $f(z) \in L$ for all $z \in \mathcal{D}$, then prove that f is constant on $\mathcal{D}$.

(2) (10 marks) Find the following integrals:

$$(i) \int_{C_1(0)} \frac{\cos z}{z} dz. \quad (ii) \int_{C_1(0)} \frac{\sin z}{z^2 - 5} dz.$$

(3) (15 marks) Let $f \in Hol(U)$, let $z_0 \in U$, and let $f'(z_0) \neq 0$. Prove that there exists a real number $r > 0$ such that

$$\frac{2\pi i}{f'(z_0)} = \int_{C_r(z_0)} \frac{1}{f(z) - f(z_0)} dz.$$

(4) (10 marks) Let $f \in Hol(B_1(0))$, and let $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1, z_2 \in B_1(0)$. Prove that there exists a scalar α such that $f(z) = \alpha z$ for all $z \in B_1(0)$.

(5) (10 marks) If $f \in Hol(\mathcal{D})$, and let f has distinct zeros $z_1, \ldots, z_n$ with multiplicities $m_1, \ldots, m_n$, respectively. Prove that there exists $g \in Hol(\mathcal{D})$ such that

$$f = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n} g,$$

on $\mathcal{D}$.

(6) (10 marks) Let $\mathcal{D}$ be a domain in $\mathbb{R}^2$, and let u and v are harmonic functions on $\mathcal{D}$. True or false (with justification): (i) $u + v$ is harmonic. (ii) uv is harmonic.

(7) (15 marks) Prove that for all polynomial $p \in \mathbb{C}[z]$,

$$\sup_{z \in C_1(0)} |z^{-1} - p(z)| \geq 1.$$

(8) (10 marks) Let f be a non-constant entire function. (i) Prove that the range of f is dense. (ii) If $|f| = 1$ on $C_1(0)$, then describe f.

(9) (15 marks) Let $\{a_n\} \subseteq \mathbb{C}$, let

$$\sum_{n=0}^{\infty} |a_n| < \infty,$$

and let

$$\sum_{n=0}^{\infty} \frac{a_n}{k^n} = 0,$$

for all integer $k \geq 2$. Prove that $a_n = 0$ for all n.